

# Pathwise construction of certain moment dualities and application to population models with balancing selection

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July 26, 2012

## Abstract

We investigate dual mechanisms for interacting particle systems. Generalizing an approach of Alkemper and Hutzenthaler in the case of coalescing duals, we show that a simple linear transformation leads to a moment duality of suitably rescaled processes. More precisely, we show how dualities of interacting particle systems of the form  $H(A, B) = q^{|A \cap B|}$ ,  $A, B \subset \{0, 1\}^N$ ,  $q \in [-1, 1)$ , are rescaled to yield moment dualities of rescaled processes. We discuss in particular the case  $q = -1$ , which explains why certain population models with balancing selection have an annihilating dual process. We also consider different values of  $q$ , and answer a question by Alkemper and Hutzenthaler.

*Keywords:* Markov processes, duality, interacting particle systems, graphical representation, annihilation, selection.

*MSC Subject classification:* 60K35.

## 1 Introduction and main result

Dualities have proved to be a powerful tool in the analysis of interacting particle systems and population models. For interacting particle systems, one generally considers two kind of duals: coalescing and annihilating duals, [Lig05, Gri79, SL95]. In connection with population models, rescaled interacting particle systems and their limits are of considerable interest, and it is natural to ask in which sense rescaling preserves dualities. Alkemper and Hutzenthaler [AH07] consider the case of coalescing dual mechanisms, and derive a ‘prototype’ moment duality under rescaling. Swart [Sw06] uses a similar idea to obtain dualities of stepping stone models. In this paper, we consider a general form of a duality for interacting particle systems, [SL95]. This includes coalescing as well as *annihilating dual mechanisms*. We prove a ‘prototype’ moment duality of linearly transformed rescaled processes, in a similar fashion as for the coalescing case, and we discuss the situation for annihilating duals in some more details. As an application, we consider branching annihilating processes and their duals. Our approach explains why population models with balancing selection generally have an annihilating dual process, as was found, for example, in [BEM07]. Since this moment duality is obtained directly from the graphical representation, our approach presents a pathwise construction. We finally discuss the connections to the Lloyd-Sudbury approach, [SL95, Sw06], which answers the question posed in [AH07].

For a Markov process  $(X_t)_{t \geq 0}$  we write  $\mathbb{P}_x$  for the law of the process started in  $x$ , and  $\mathbb{E}_x$  for the corresponding expectation. Two Markov processes  $(X_t)_{t \geq 0}, (Y_t)_{t \geq 0}$  with state spaces  $E$  and  $F$ , respectively, are called *dual* with respect to the *duality function*  $H : E \times F \rightarrow \mathbb{R}$  if for all  $t \geq 0$ ,  $x \in E$ ,  $y \in F$  the equality

$$\mathbb{E}_x[H(X_t, y)] = \mathbb{E}_y[H(x, Y_t)] \quad (1)$$

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holds. This means that the long-term behaviour of one process is – to some extent – determined by the long-term behaviour of the other process. The usefulness of a duality depends on the duality function  $H$ . If for example  $(X_t)$  takes values in  $\mathbb{R}$ , and  $(Y_t)$  in  $\mathbb{N}$ , we call a duality with respect to the function

$$H(x, y) = x^y$$

a *moment duality*, since it determines all the moments of  $X_t$ . For practical purposes, it is often useful to have a *pathwise* construction of the dual processes, which means that they can be constructed on the same probability space in some explicit way, for example as functions of one underlying driving process. In the case of interacting particle systems, this construction is usually provided by the *graphical representation*, [Har78, Gri79]. We explain this below in the setup that we use for the present paper.

Let  $N \in \mathbb{N}$ , and let  $E_N := \{0, 1\}^N$ . We write  $x \in E_N$  as a vector  $x = (x_i)_{1 \leq i \leq N}$ . A partial order on  $E_N$  is given by  $x \leq y \Leftrightarrow x_i \leq y_i \forall 1 \leq i \leq N$ . We write  $x \wedge y$  for the minimum of  $x$  and  $y$  with respect to this ordering. Let  $(X_t^N)_{t \geq 0}$  and  $(Y_t^N)_{t \geq 0}$  denote Markov processes defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with values in  $E_N$ ,  $X_t^N = (X_t^N(i))_{i=1, \dots, N}$ . Let  $A_t^N := \{i : X_t^N(i) = 1\}$  and  $B_t^N := \{i : Y_t^N(i) = 1\}$ ; this defines processes taking values in the subsets of  $\{1, \dots, N\}$ . We write  $|X_t^N| := |A_t^N|$  for the cardinality of the set  $A_t^N$ , that is for the number of 1's. Sudbury and Lloyd [SL95] argue that in this context, duality functions that are functions of  $A \cap B$  alone should be of the form

$$H(A, B) = q^{|A \cap B|}, \quad A, B \subset \{1, \dots, N\},$$

for some  $q \in \mathbb{R} \setminus \{1\}$ . We take this as a motivation to say that two  $E_N$ -valued Markov processes  $(X_t^N), (Y_t^N)$  are  $q$ -dual if

$$\mathbb{E}_x[q^{|X_t^N \wedge Y_0^N|}] = \mathbb{E}_y[q^{|X_0^N \wedge Y_t^N|}] \quad \forall x, y \in E_N, t \geq 0. \quad (2)$$

That is, the duality function is  $H(x, y) = q^{|x \wedge y|}$ . Special cases are  $q = 0$ , which is called *coalescing duality*, and  $q = -1$ , which is called *annihilating duality*. In these cases the duality function becomes  $0^{|x \wedge y|} = 1_{\{x \wedge y = 0\}}$ , and  $(-1)^{|x \wedge y|} = 1 - 2 \times 1_{\{|x \wedge y| \text{ is odd}\}}$ , respectively.

We now describe the graphical representation for such dualities. This is classical, for more details see [Har78, Gri79]. For each  $i \in \{1, \dots, N\}$ , draw a vertical line of length  $T$ , which represents time up to a finite end point  $T$ . We consider ordered pairs  $(i, j)$  with  $i, j \in \{1, \dots, N\}$ . For each such pair, run  $m \in \mathbb{N}$  independent Poisson processes with parameters  $(\lambda_{ij}^k), k = 1, \dots, m$ . At the time of an arrival draw an arrow from the line corresponding to  $i$  to the line corresponding to  $j$ , marked with the index  $k$  of the process. Do this independently for each ordered pair, for each  $k = 1, \dots, m$ . For each  $k$ , we define functions  $f^k, g^k : \{0, 1\}^2 \rightarrow \{0, 1\}^2$ . A Markov process  $(X_t^N)$  with càdlàg paths is then constructed by specifying an initial condition  $x = (x_i)_{i=1, \dots, N}$ , and the following dynamics:  $X_t^N = x$  until the time of the first arrow in the graphical representation. If this arrow points from  $i$  to  $j$  and is labelled  $k$ , then the pair  $(x_i, x_j)$  is changed to  $f^k(x_i, x_j)$ , and the other coordinates remain unchanged. Go on until the next arrow, and proceed exactly in the same way. The dual process  $(Y_t^N)$  is constructed using the same Poisson processes, but started at the final time  $T > 0$ , running time backwards, inverting the order of all arrows, and using the functions  $g^k$  instead of  $f^k$ .

Following [AH07], we call the functions  $f^k, g^k$  *basic mechanisms*, and we generalize the definition of dual basic mechanisms given by Alkemper and Hutzenthaler. For  $x = (x_1, x_2) \in \{0, 1\}^2$  we use the notation  $x^\dagger := (x_2, x_1)$ ; the dagger accounts for the reversal of an arrow.

**Definition 1.1.** *Two basic mechanisms  $f, g : \{0, 1\}^2 \rightarrow \{0, 1\}^2$  are called  $q$ -dual mechanisms if and only if*

$$q^{|x \wedge (g(y^\dagger))^\dagger|} = q^{|f(x) \wedge y|} \quad \forall x, y \in \{0, 1\}^2. \quad (3)$$

It is clear that two processes constructed using  $q$ -dual mechanisms are  $q$ -dual processes.

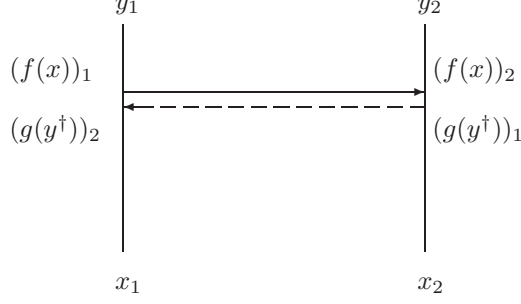


FIGURE 1

**Lemma 1.2.** Fix  $m \in \mathbb{N}$ ,  $q \in \mathbb{R} \setminus \{1\}$  and  $T > 0$ . For every  $k = 1, \dots, m$ , let  $f^k, g^k$  be  $q$ -dual basic mechanisms. Let  $(X_t^N), (Y_t^N)$  be Markov processes with state space  $E_N$ , constructed using the mechanisms  $f_k$ ,  $k = 1, \dots, m$  and  $g_k$ ,  $k = 1, \dots, m$ , respectively, and Poisson processes with the same symmetric parameters  $\lambda_{ij}^k = \lambda_{ji}^k$ ,  $k = 1, \dots, m$ ,  $i, j = 1, \dots, N$ . Then there exists a process  $(\hat{Y}_t^N)$  such that

$$\hat{Y}_t^N \stackrel{d}{=} Y_t^N \quad \text{and} \quad q^{|X_t \wedge \hat{Y}_t|} = q^{|X_0 \wedge \hat{Y}_t|} \quad \text{a.s. } \forall 0 \leq t \leq T. \quad (4)$$

*Proof.* Since we assume that the Poisson processes have the same rates, we can construct  $\hat{Y}_t^N$  from the graphical representation of  $(X_t^N)$ , using the same realization of the Poisson processes, reversing time and the directions of all the arrows. It is clear from the construction that then  $\hat{Y}_t^N \stackrel{d}{=} Y_t^N$ , and  $q^{|X_t \wedge \hat{Y}_0|} = q^{|X_0 \wedge \hat{Y}_t|}$  hold (see Figure 1). For some more details, in the case of coalescing mechanisms, compare the proof of Proposition 2.3 of [AH07].  $\square$

Taking expectations, the following is then obvious.

**Corollary 1.3.** In the situation of Lemma 1.2, the processes  $(X_t^N)$  and  $(Y_t^N)$  are  $q$ -dual.

*Remark 1.4.* We note that Lemma 1.2 tells us that  $(X_t^N)$  and  $(Y_t^N)$  are dual in a very strong sense, namely, the equation  $H(X_t^N, \hat{Y}_t^N) = H(X_0^N, \hat{Y}_t^N)$  holds almost surely instead of just in expectation. We call such processes *strongly pathwise dual*. We have just seen that a construction via graphical representation and  $q$ -dual basic mechanisms automatically leads to a strong pathwise duality.

We are now ready to state and prove the main result of this article.

**Theorem 1.5.** Let  $(X_t^N), (Y_t^N)$  be Markov processes with state space  $E_N$  that are  $q_N$ -dual for some  $q_N \in [-1, 1)$ . Choose exchangeable initial conditions  $X_0^N, Y_0^N \in E_N$ , fixing  $|X_0^N| = k_N, |Y_0^N| = n_N$ , and suppose that  $X_t^N$  and  $Y_t^N$  stay exchangeable for all  $t > 0$ . Assume that  $n_N/N \rightarrow 0$  and  $\mathbb{E}[|Y_{t_N}^N|/N] \rightarrow 0$  as  $N \rightarrow \infty$ , for some time scale  $t_N \geq 0$ . Then

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \left( 1 + (q_N - 1) \frac{|X_0^N|}{N} \right)^{|Y_{t_N}^N|} \right] = \lim_{N \rightarrow \infty} \mathbb{E} \left[ \left( 1 + (q_N - 1) \frac{|X_{t_N}^N|}{N} \right)^{|Y_0^N|} \right],$$

provided that the limits exist.

Theorem 1.5 applies, for example, to processes constructed from basic mechanisms with rates  $\lambda_{ij}^k$  that do not depend on  $i$  and  $j$ ; all our later examples fall into this class. As illustrated by the Moran model, however, exchangeability of the arrows is not a necessary condition for the processes to stay exchangeable.

Depending on the scaling, Theorem 1.5 may lead to a moment duality, if  $\frac{|X_{t_N}^N|}{N} \rightarrow X_t$ , and  $|Y_{t_N}^N| \rightarrow Y_t$ , as we then get  $\mathbb{E}[(1 + (q - 1)X_0)Y_t] = \mathbb{E}[(1 - (q - 1)X_t)Y_0]$ . If  $X^N$  and  $Y^N$  have the same scaling, we may get a Laplace duality, see [AH07] for an example.

*Proof.* The proof relies on the simple fact that the distribution of  $|X \wedge Y|$  given  $|X|$  and  $|Y|$  is approximately binomial with parameters  $|Y|$  and  $\frac{|X|}{N}$ , provided that  $|Y|$  is small with respect to  $N$ . Indeed,

$|X \wedge Y|$  follows a hypergeometric distribution, since it is obtained by distributing the  $|Y|$  1's of the  $Y$ -configuration onto the  $|X|$  1's of the  $X$ -configuration, without hitting the same 1 twice. Approximating the hypergeometric distribution by a binomial distribution will give us the result. Let  $Z^N \sim \text{Bin}(n_N, \frac{x_N}{N})$  with  $x_N \in \{0, \dots, N\}$  and  $n_N/N \rightarrow 0$ . By Theorem 4 of [DF80], we can bound the total variation distance between the hypergeometric and the binomial distribution as

$$\|\text{Hyp}(N, x_N, n_N) - \text{Bin}(n_N, \frac{x_N}{N})\|_{\text{TV}} \leq \frac{4n_N}{N}.$$

Since we assumed  $q_N \in [-1, 1)$ , we obtain

$$\begin{aligned} \mathbb{E} \left[ q_N^{|X_{t_N}^N \wedge Y_0^N|} \mid |X_{t_N}^N| = x_N, |Y_0^N| = n_N \right] &= \sum_{k=0}^{n_N} q_N^k \mathbb{P}(|X_t^N \wedge Y_0| = k \mid |X_t^N| = x_N, |Y_0^N| = n_N) \\ &= \mathbb{E} \left[ q_N^{Z^N} \right] + o(1), \end{aligned}$$

where  $\mathbb{E}[q_N^{Z^N}]$  is just the probability generating function of the binomial variable  $Z^N$ . This is well known to be

$$\mathbb{E} \left[ q_N^{Z^N} \right] = \left( q_N \frac{x_N}{N} + \left( 1 - \frac{x_N}{N} \right) \right)^{n_N} = \left( 1 + (q_N - 1) \frac{x_N}{N} \right)^{n_N}.$$

Averaging over the initial conditions  $X_0^N$  with  $|X_0^N| = k_N$ , we obtain

$$\mathbb{E} \left[ q_N^{|X_t^N \wedge Y_0^N|} \right] = \mathbb{E} \left[ \left( 1 + (q_N - 1) \frac{|X_{t_N}^N|}{N} \right)^{|Y_0^N|} \right] + o(1).$$

In the same way, using  $\mathbb{E}[|Y_t^N|]/N \rightarrow 0$ , we get

$$\mathbb{E} \left[ q_N^{|X_0^N \wedge Y_{t_N}^N|} \right] = \mathbb{E} \left[ \left( 1 + (q_N - 1) \frac{|X_0^N|}{N} \right)^{|Y_{t_N}^N|} \right] + o(1).$$

By duality,

$$\mathbb{E} \left[ \left( 1 + (q_N - 1) \frac{|X_{t_N}^N|}{N} \right)^{|Y_0^N|} \right] = \mathbb{E} \left[ \left( 1 + (q_N - 1) \frac{|X_0^N|}{N} \right)^{|Y_{t_N}^N|} \right] + o(1).$$

Letting  $N \rightarrow \infty$  gives the desired result.  $\square$

For the binomial approximation, it was necessary to assume that  $Y^N/N \rightarrow 0$ . We now give a result for the case that both  $X_t^N$  and  $Y_t^N$  scale with  $N$ .

**Proposition 1.6.** *Let  $(X_t^N), (Y_t^N)$  be Markov processes with state space  $E_N$  that are  $q_N$ -dual for some  $q_N$  such that  $\lim_{N \rightarrow \infty} N(q_N - 1) = -\lambda \in (-\infty, 0]$ . Choose exchangeable initial conditions  $X_0^N, Y_0^N \in E_N$  and suppose that both processes stay exchangeable at  $t > 0$ . Assume that the process  $\frac{|Y_t^N|}{N}$  converges weakly to some process  $\tilde{Y}_t$ , that  $\frac{|X_t^N|}{N}$  converges weakly to  $\tilde{X}_t$ . Then  $(\tilde{X}_t)$  and  $(\tilde{Y}_t)$  are dual with respect to*

$$H(x, y) = e^{-\lambda xy}.$$

*Proof.* We have

$$\begin{aligned} \mathbb{E} \left[ q_N^{|X_t^N \wedge Y_0^N|} \right] &= \mathbb{E} \left[ q_N^{\sum_{i=1}^N X_t^N(i) Y_0^N(i)} \right] \\ &= \mathbb{E} \left[ \left( 1 + \frac{N(q_N - 1)}{N} \right)^{\sum_{i=1}^N X_t^N(i) Y_0^N(i)} \right] \rightarrow \mathbb{E} \left[ e^{-\lambda \tilde{X}_t \tilde{Y}_0} \right], \end{aligned}$$

since by exchangeability,  $\frac{1}{N} \sum_{i=1}^N X_t^N(i) Y_0^N(i) \rightarrow \tilde{X}_t \tilde{Y}_0$ .  $\square$

*Remark 1.7.* Note that for these results we only assume *duality* of the processes, and not necessarily strong pathwise duality in the sense of (4). However, if we start from a graphical representation, this result provides a pathwise construction of the limiting moment duality; see Section 3 for a related discussion.

In the remainder of the paper, we discuss in some detail the case of annihilating duals and possible dual mechanisms. We restate Theorem 1.5 in this particular case, and discuss several examples where this result can be applied to rederive certain dualities, mostly known in the literature. The examples illuminate in particular the connection between annihilating duals and population models with balancing selection, as studied for example in [BEM07]. In the last section we consider different values of  $q$ . The last example answers an open question of [AH07] concerning a self-duality derived in [AS05].

## 2 Annihilating duality

### 2.1 Annihilating dual mechanisms

In this section, we discuss the special case of a  $q$ -duality with  $q = -1$ , which is an *annihilating duality*. This duality relation can be written as

$$\mathbb{P}_x(|X_t^N \wedge Y_0^N| \text{ is odd}) = \mathbb{P}_y(|X_0^N \wedge Y_t^N| \text{ is odd}) \quad (5)$$

for all  $x, y \in E_N$ . In order to apply our rescaling result, we identify some basic mechanisms which lead to annihilating dualities. It is interesting to compare them to some of the coalescing mechanisms. In the following table, we give a list of the mechanisms that we are interested in, and afterwards discuss their duality relations.

	$f(0,0)$	$f(0,1)$	$f(1,0)$	$f(1,1)$	
$f^R$	(0,0)	(0,0)	(1,1)	(1,1)	resampling
$f^C$	(0,0)	(0,1)	(0,1)	(0,1)	walk-coalescence
$f^A$	(0,0)	(0,1)	(0,1)	(0,0)	walk-annihilation
$f^D$	(0,0)	(0,0)	(0,1)	(0,1)	death-walk
$f^{BC}$	(0,0)	(0,1)	(1,1)	(1,1)	branching-coalescence
$f^{BA}$	(0,0)	(0,1)	(1,1)	(1,0)	branching-annihilation

The names given to the mechanisms are chosen to suggest an interpretation. In the resampling mechanism, the first position gives its type (0 or 1) to the second one. In the death-walk mechanism, a particle in the second position dies, after which a particle in the first position walks to the second position. Walk mechanisms suggest that a particle in the first position jumps to the second position, and either coalesces or annihilates if there is a particle present. In branching mechanisms, a particle in the first position produces a new particle in the second position, which either coalesces or annihilates with a particle already present.

*Remark 2.1* (Coalescing duals). In [AH07], the coalescing dual mechanisms were classified. Concerning the dualities given in the above table, the following *coalescing* dualities were established: (i)  $f^R$  and  $f^C$  are coalescing duals, (ii)  $f^D$  is a coalescing self-dual, and (iii)  $f^{BC}$  is a coalescing self-dual.

**Lemma 2.2.** (a) *Two basic mechanisms  $f, g$  are annihilating dual mechanisms if and only if*

$$|x \wedge (g(y^\dagger))^\dagger| \text{ is odd} \Leftrightarrow |f(x) \wedge y| \text{ is odd}.$$

(b) *With the notation of the above table, we have the following:*

- (i)  $f^R$  and  $f^A$  are annihilating duals
- (ii)  $f^D$  is an annihilating self-dual
- (iii)  $f^{BA}$  is an annihilating self-dual.

*Proof.* (a) is obvious. We verify (b) using the table of basic mechanisms. (i) We have that  $f^R(x) \wedge y$  is odd if and only if  $x = (1, 0)$  or  $x = (1, 1)$ , and  $y = (0, 1)$  or  $(1, 0)$ . In both cases,  $(f^A(y^\dagger))^\dagger = (1, 0)$ , and  $(1, 0) \wedge x$  is odd if and only if  $x \in \{(1, 0), (1, 1)\}$ . By (a) this proves the duality of  $f^R$  and  $f^A$ . For (ii) note that  $f^D(x) \wedge y$  is odd if and only if  $x \in \{(1, 0), (1, 1)\}$  and  $y \in \{(0, 1), (1, 1)\}$ . But then  $(f^D(y^\dagger))^\dagger = (1, 0)$ , and the claim follows. (iii) For  $f^{BA}(x) \wedge y$  to be odd we need  $x = (0, 1)$  and  $y \in \{(0, 1), (1, 1)\}$ , or  $x = (1, 0)$  and  $y \in \{(0, 1), (1, 0)\}$ , or  $x = (1, 1)$  and  $y \in \{(1, 0), (1, 1)\}$ . In all cases  $(f^{BA}(y^\dagger))^\dagger \wedge x$  is odd, and there are no other possibilities.  $\square$

*Remark 2.3.* The list of duals is not complete. For a full classification of coalescing duals see [AH07]. Note that  $f^R$  and  $f^D$  have both a coalescing and an annihilating dual process. The death-walk-mechanism  $f^D$  is  $q$ -self-dual for any  $q \in \mathbb{R}$ : From the table of dual mechanisms one can check that  $|x \wedge (f^D(y^\dagger))^\dagger| = |f^D(x) \wedge y|$  for all  $x, y \in \{0, 1\}^2$ .

*Remark 2.4.* It is easy to see that  $q$ -dual mechanisms always satisfy  $f(0, 0) = (0, 0)$ . However, unlike the case of coalescing duality, a mechanism need not be monotone in order to have an annihilating dual, as can be seen from the self-duality of the branching-annihilating mechanism.

We can now restate our Theorem 1.5 in the special case of annihilating duals. This special case is motivated by the observation, made in [BEM07], that a particular model of populations with balancing selection, after a transformation of the form  $x \mapsto 1 - 2x$ , is dual to a double-branching annihilating process. Our result shows why this transformation occurs in annihilating processes. A non-spatial version of this model will be discussed as an example in the following section.

**Corollary 2.5.** *Let  $(X_t^N), (Y_t^N)$  be Markov processes with state space  $E_N$  such that  $\mathbb{P}_x(|X_t^N \wedge Y_0^N| \text{ is odd}) = \mathbb{P}_y(|X_0^N \wedge Y_t^N| \text{ is odd})$  holds for all  $x, y \in E_N$ . Let  $k_N, n_N \in \mathbb{N}$ , and choose exchangeable initial conditions  $x_N, y_N \in E_N$  such that  $|x_N| = k_N, |y_N| = n_N$ . Suppose that both processes stay exchangeable at  $t > 0$ , and assume that  $n_N/N \rightarrow 0$  and  $\mathbb{E}[|Y_{t_N}^N|/N] \rightarrow 0$  as  $N \rightarrow \infty$ . Then*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \left( 1 - \frac{2|X_0^N|}{N} \right)^{|Y_{t_N}^N|} \right] = \lim_{N \rightarrow \infty} \mathbb{E} \left[ \left( 1 - \frac{2|X_{t_N}^N|}{N} \right)^{|Y_0^N|} \right],$$

*provided that the limits exist.*

As before, assuming that a limiting process  $(p_t)$  of  $1 - \frac{2|X_{t_N}^N|}{N}$  and  $n_t$  of  $|Y_{t_N}^N|$  exists, these processes satisfy the moment duality

$$\mathbb{E}_n[p_0^{n_t}] = \mathbb{E}_p[p_t^{n_0}].$$

*Proof.* Corollary 2.5 is a consequence of Theorem 1.5, by setting  $q = -1$ . It can also be understood from the fact that the probability that a binomial random variable with parameters  $n, p$  takes an odd value is given by  $\frac{1}{2}(1 - (1 - 2p)^n)$ . Then we have, by binomial approximation,

$$\mathbb{P}(|X_t^N \wedge Y_0^N| \text{ is odd} \mid |X_t^N| = x_N, |Y_0^N| = n_N) = \frac{1}{2} \left( 1 - \left( 1 - \frac{2x_N}{N} \right)^{n_N} \right) + o(1),$$

as in the proof of Theorem 1.5; again duality, averaging over the exchangeable initial conditions, and taking limits, gives the result.  $\square$

## 2.2 Examples

In this section we derive some (mostly well-known) dualities by rescaling dualities of interacting particle systems. We will assume that the following mechanisms occur in the process  $(X_t^N)$ :  $f^R$  occurs with rate  $\frac{r_N}{N}$  for each ordered pair  $(i, j)$ ,  $i, j \in \{1, \dots, N\}$ ,  $f^C$  with rate  $\frac{c_N}{N}$ ,  $f^A$  with rate  $\frac{a_N}{N}$ ,  $f^D$  with rate  $\frac{d_N}{N}$ ,  $f^{BA}$  with rate  $\frac{b_N^a}{N}$ , and  $f^{BC}$  with rate  $\frac{b_N^c}{N}$ . Moreover, set  $b_N := b_N^a + b_N^c$ . Consider the process  $|X_t^N|$  taking values in  $\{0, \dots, N\}$ . Note that if  $|X_t^N| = k$ , then the number of ordered pairs of certain types is easily computed: The number of  $(0, 1)$ -pairs (or equivalently of  $(1, 0)$ -pairs) is

equal to  $k(N - k)$ , the number of  $(1, 1)$ -pairs is equal to  $k(k - 1)$ . Hence, the process  $|X_t^N|, t \geq 0$ , makes the following transitions:

$$k \rightarrow k + 1 \quad \text{at rate} \quad \frac{r_N + b_N}{N} k(N - k), \quad (6)$$

$$k \rightarrow k - 1 \quad \text{at rate} \quad \frac{r_N + d_N}{N} k(N - k) + \frac{c_N + d_N + b_N^a}{N} k(k - 1), \quad (7)$$

$$k \rightarrow k - 2 \quad \text{at rate} \quad \frac{a_N}{N} k(k - 1). \quad (8)$$

In the next sections, we will consider processes of this type and their duals for various values and scalings of the rates.

### 2.2.1 Branching annihilating process

Let  $a_N = d_N = b_N^c = 0$ , and assume  $\frac{r_N}{N} \rightarrow \alpha \geq 0$  and  $b_N = b_N^a \rightarrow \beta \geq 0$ , as  $N \rightarrow \infty$ . The different scaling of the mechanism is interpreted in the sense that in the limit, the resampling affects pairs of particles, while branching happens at a fixed rate per single particle. The rescaled discrete process  $\frac{|X_t^N|}{N}$  has, according to (6) and (7), discrete generator

$$\begin{aligned} \tilde{G}_N f\left(\frac{k}{N}\right) &= \frac{r_N}{N} k(N - k) \left( f\left(\frac{k+1}{N}\right) + f\left(\frac{k-1}{N}\right) - 2f\left(\frac{k}{N}\right) \right) \\ &\quad + \frac{b_N}{N} k(k - 1) \left( f\left(\frac{k-1}{N}\right) - f\left(\frac{k}{N}\right) \right) + \frac{b_N}{N} k(N - k) \left( f\left(\frac{k+1}{N}\right) - f\left(\frac{k}{N}\right) \right). \end{aligned}$$

Assume now  $\frac{k}{N} \rightarrow x$  as  $N \rightarrow \infty$  and  $f$  twice differentiable. Then, noting  $\lim_{N \rightarrow \infty} N \left( f\left(\frac{k+1}{N}\right) - f\left(\frac{k}{N}\right) \right) = f'(x)$  and  $\lim_{N \rightarrow \infty} N^2 \left( f\left(\frac{k+1}{N}\right) + f\left(\frac{k-1}{N}\right) - 2f\left(\frac{k}{N}\right) \right) = f''(x)$ , we see that  $\tilde{G}_N f(k/N)$  converges to

$$\tilde{G}f(x) = \beta x(1 - 2x)f'(x) + \alpha x(1 - x)f''(x),$$

which is the generator of the one-dimensional diffusion given by the SDE

$$dX_t = \beta X_t(1 - 2X_t)dt + \sqrt{2\alpha X_t(1 - X_t)}dB_t.$$

This is a Wright-Fisher diffusion with local drift  $\beta x(1 - 2x)$ . The drift has the effect of pushing  $X_t$  towards the values 0 and 1/2 and may be interpreted as a selection promoting heterozygosity – this interpretation will become more evident in the next example. Note that it is not difficult to incorporate death as well: If  $d_N \rightarrow \delta > 0$ , the resulting diffusion reads

$$dX_t = \beta X_t(1 - X_t)dt - \delta X_t dt + \sqrt{\alpha X_t(1 - X_t)}dB_t.$$

Consider now the dual process. According to Lemma 2.2,  $(Y_t^N)$  where  $f^A$  happens at rate  $\frac{r_N}{N}$ ,  $f^{BA}$  at  $\frac{b_N}{N}$  is an annihilating dual of  $(X_t^N)$ . The generator of  $|Y_t^N|$  is

$$\begin{aligned} G_N f(k) &= \frac{b_N}{N} k(N - k) (f(k + 1) - f(k)) + \frac{b_N}{N} k(k - 1) (f(k - 1) - f(k)) \\ &\quad + \frac{r_N}{N} k(k - 1) (f(k - 2) - f(k)). \end{aligned}$$

As  $N \rightarrow \infty$ , when  $f(n) \rightarrow 0$  fast enough as  $n \rightarrow \infty$ , this converges to

$$Gf(k) := \beta k (f(k + 1) - f(k)) + \alpha k(k - 1) (f(k - 2) - f(k)),$$

which is the generator of a branching annihilating process on  $\mathbb{N}_0$ . Including death, we get

$$Gf(k) := \beta k (f(k + 1) - f(k)) + \alpha k(k - 1) (f(k - 2) - f(k)) + \delta k (f(k - 1) - f(k)).$$

Weak convergence of the rescaled processes is obtained from the convergence of the generators by classical theory, see for example [EK]. By Corollary 2.5 we obtain for the limiting processes  $(X_t), (Y_t)$  the duality

$$\mathbb{E}_x [(1 - 2X_t)^{Y_0}] = \mathbb{E}_y [(1 - 2X_0)^{Y_t}].$$

*Remark 2.6.* Write  $p_t := 1 - 2X_t$ . Itô's formula yields  $dp_t = \beta(p_t^2 - p_t)dt - \sqrt{\alpha(1 - p_t^2)}dB_t$ , from which – at least heuristically – it is easy to read off the moment duality of the process  $(p_t)_{t \geq 0}$  and the branching annihilating process. Our method establishes this duality in a straightforward manner, and also shows why the transformation  $p_t = 1 - 2X_t$  has to be applied.



### 2.2.2 Double-branching annihilating process and populations with balancing selection

One of our motivations was to understand the transformation  $x \mapsto 1 - 2x$  applied in [BEM07] in order to obtain the duality between the competing species model and double-branching annihilating random walk, which is parity preserving. This is substantially different from our last example, as a branching event produces *two* new particles and not one, which is not taken care of by our basic mechanisms. However, it is easily implemented if we allow for multiple arrows in the graphical representation, or, equivalently, for basic mechanisms  $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$ .

Assume that for each ordered pair  $(i, j)$  the  $f^A$ -mechanism happens at rate  $\frac{a_N}{N}$ , and construct an additional mechanism  $f$  in the following way: For each ordered triple  $(i, j, k)$ ,  $i, j, k = 1, \dots, N$ , draw, at rate  $\frac{b_N}{N^2}$ , an arrow from  $i$  to  $j$  and from  $i$  to  $k$ . Then, if an arrow is encountered, a transition  $f^{BA}$  occurs for the two pairs  $(i, j)$  and  $(i, k)$ . This means that at such a double transition, the state of the triple  $(x_i, x_j, x_k)$  is changed according to the following table:

$x$	(000)	(001)	(010)	(100)	(101)	(110)	(011)	(111)
$f(x)$	(000)	(001)	(010)	(111)	(110)	(101)	(011)	(100)

The dual mechanism  $\tilde{f}$  is given by inverting the arrows and applying the dual mechanism  $f^{BA}$  to each of the two arrows, that is, to the pairs  $(x_j, x_i)$  and  $(x_k, x_i)$  with the additional rule that two 1's at the same place annihilate each other, that is, given by the table

$x$	(000)	(001)	(010)	(100)	(101)	(110)	(011)	(111)
$\tilde{f}(x)$	(000)	(101)	(110)	(100)	(001)	(010)	(011)	(111)

It is clear that these two mechanisms are annihilating dual mechanisms, either by direct verification, or by noting that the double-branching transition is the result of two  $f^{BA}$ -transitions happening one right after the other, cf. Figure 2 for a graphical representation.

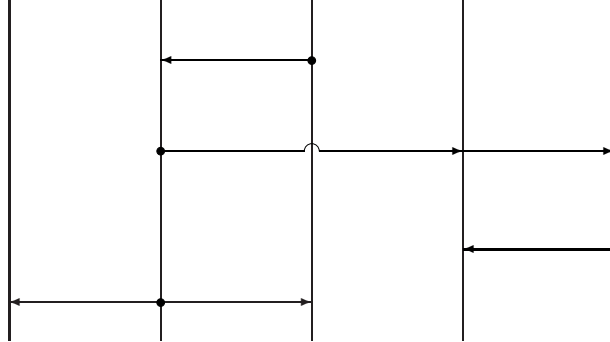


FIGURE 2

Let now  $(Y_t^N)$  be the process constructed from the graphical representation, where  $f^A$  happens at rate  $\frac{a_N}{N}$ , and  $f$  at rate  $\frac{b_N}{N^2}$ . Then  $|Y_t^N|$  has the transitions

$$\begin{aligned} k \rightarrow k+2 & \quad \text{at rate} \quad \frac{b_N}{N^2} k(N-k)(N-k-1), \\ k \rightarrow k-2 & \quad \text{at rate} \quad \frac{b_N}{N^2} k(k-1)(k-2) + \frac{a_N}{N} k(k-1), \end{aligned}$$

since  $k(N-k)(N-k-1)$  is the number of (100)-triples if there are  $k$  1's, etc. Assume  $\frac{a_N}{N} \rightarrow \alpha$  and  $b_N \rightarrow \beta$  as  $N \rightarrow \infty$ . Then the generator of  $|Y_t^N|$  converges to

$$Gf(k) = \beta k(f(k+2) - f(k)) + \alpha k(k-1)(f(k-2) - f(k)),$$

which is the generator of a double-branching annihilating process. For the dual process  $(X_t^N)$ , with mechanisms  $f^R$  and  $\tilde{f}$ , we obtain the transitions

$$\begin{aligned} k \rightarrow k+1 & \quad \text{at rate} \quad \frac{b_N}{N^2} 2k(N-k)(N-k-1) + \frac{a_N}{N} k(N-k), \\ k \rightarrow k-1 & \quad \text{at rate} \quad \frac{b_N}{N^2} 2k(k-1)(N-k) + \frac{a_N}{N} k(N-k), \end{aligned}$$



which yield for  $N \rightarrow \infty$ , if  $\frac{k}{N} \rightarrow x$ ,

$$\tilde{G}f(x) = 2\beta x(1-x)(1-2x)f'(x) + \alpha x(1-x)f''(x).$$

$\tilde{G}$  is exactly the generator of the non-spatial version of the competing species model of [BEM07], given by the SDE

$$dX_t = 2\beta X_t(1-X_t)(1-2X_t)dt + \sqrt{2\alpha X_t(1-X_t)}dB_t.$$

For a motivation of this model as well as results on the long-term behaviour of its spatial version, see [BEM07].

### 3 Other values of $q$ and self-duality of the resampling-selection process

Alkemper and Hutzenthaler [AH07] ask whether the self-duality derived in [AS05] for the so-called resampling-selection process

$$dX_t = sX_t(1-X_t)dt - mX_tdt + \sqrt{2rX_t(1-X_t)}dB_t \quad (9)$$

could be constructed pathwise using the approach of dual basic mechanisms. This question can be split into several distinct questions. First, is it possible to construct  $(X_t)$  as the scaling limit of self-dual processes  $(X_t^N)$  for interacting particle systems in such a way that the self-duality of  $(X_t)$  is inherited from the self-duality of  $(X_t^N)$  (compare [Sw06])? Second, is it possible to explain the self-duality of the discrete process  $(X_t^N)$  using a pathwise construction? Third, can we choose to construct the discrete processes with  $q$ -dual basic mechanisms, thus obtaining interacting particle systems that are strongly pathwise dual?

As we shall see, the answer to the first two questions is yes: There is a pathwise construction, using the basic mechanisms from Section 2, yielding  $q$ -dual processes  $(X_t^N)$  and  $(Y_t^N)$  that rescale to resampling-selection processes. In general, however, our basic mechanisms will not be  $q$ -dual in the sense of Definition 1.1; the third question remains, therefore, open.

The  $q$ -duality of our interacting particle systems will not follow from  $q$ -duality of basic mechanisms but, instead, from a criterion derived in [SL95], applied in a context similar to ours in [Sw06]. Let us first explain how this criterion applies to processes constructed from our basic mechanisms. Sudbury and Lloyd consider Markov processes on  $\{0, 1\}^\Lambda$ , for some graph  $\Lambda$ , whose generator is of the form

$$\begin{aligned} \bar{G}f(x) = \sum_{i \neq j} \bar{q}(i, j) & \left( \frac{\bar{a}}{2} x(i)x(j)(f(x - \delta_i - \delta_j) - f(x)) + \bar{b}x(i)(1-x(j))(f(x + \delta_j) - f(x)) \right. \\ & + \bar{c}x(i)x(j)(f(x - \delta_i) - f(x)) + \bar{d}x(i)(1-x(j))(f(x - \delta_i) - f(x)) \\ & \left. + \bar{e}x(i)(1-x(j))(f(x - \delta_i + \delta_j) - f(x)) \right), \quad x \in \{0, 1\}^\Lambda \end{aligned}$$

with non-negative parameters  $\bar{a}, \dots, \bar{e}$ , and  $\bar{q}(i, j)$  defined as follows. When  $i$  and  $j$  are neighbors in  $\Lambda$  (meaning that they are connected by an edge in the graph), then  $\bar{q}(i, j) = 1/N_i$ , with  $N_i$  the number of neighbors of  $i$ ; when  $i$  and  $j$  are not neighbors,  $\bar{q}(i, j) = 0$ . Thus when  $\Lambda$  is the complete graph on  $\{0, 1, \dots, N\}$ ,  $\bar{q}(i, j) = 1/N$  for all  $i \neq j$ . The letters  $\bar{a}, \bar{b}, \bar{c}, \bar{d}$  and  $\bar{e}$  refer to annihilation, branching, coalescence, death and exclusion.

We are interested in Markov processes with state space  $\{0, 1\}^{\{0, \dots, N\}}$  constructed from the basic mechanisms of Section 2 and rates chosen as follows: For every pair  $(i, j)$ , the mechanisms  $f^A$ ,  $f^{BA}$ ,  $f^{BC}$ ,  $f^C$ ,  $f^D$  and  $f^R$  happen at rates  $a/N$ ,  $b^a/N$ ,  $b^c/N$ ,  $c/N$ ,  $d/N$  and  $r/N$ . The infinitesimal generator of this process is of the Sudbury-Lloyd form (3) with

$$\bar{a} = 2a, \quad \bar{b} = b^a + b^c + r, \quad \bar{c} = b^a + c + d, \quad \bar{d} = d + r, \quad \bar{e} = a + c + d \quad (10)$$

and  $\bar{q}(i, j) = 1/N$  for all  $i \neq j$ . We shall refer to this process as the process obtained from the basic mechanisms via the rate parameters  $a$ ,  $b^a$ ,  $b^c$ ,  $d$  and  $r$ . Note that not every Sudbury-Lloyd process can be constructed with our basic mechanisms: for example, if  $2\bar{e} < \bar{a}$ , any solution of Eq. (10) has negative

rate parameters  $c < 0$  or  $d < 0$ . Furthermore, the construction is not unique – there are in general several choices of rate parameters  $a, \dots, r$  leading to the same Sudbury-Lloyd process.

Sudbury and Lloyd give several conditions for  $q$ -duality of their models. A concise formula is [S00, Equation (9)], which in our notation reads

$$\bar{a}' = \bar{a} + 2q\gamma, \quad \bar{b}' = \bar{b} + \gamma, \quad \bar{c}' = \bar{c} - (1+q)\gamma, \quad \bar{d}' = \bar{d} + \gamma, \quad \bar{e}' = \bar{e} - \gamma \quad (11)$$

where  $\gamma = (\bar{a} + \bar{c} - \bar{d} + \bar{b}q)/(1-q)$ . Eq. (11) is easily translated into a criterion for processes obtained from our basic mechanisms.

**Proposition 3.1.** *Let  $(X_t), (Y_t)$  be the Sudbury-Lloyd processes obtained from our basic mechanisms with respective rate parameters  $a, b^a, b^c, c, d, r$  and  $a', b^{a'}, b^{c'}, c', d', r'$ . Then*

1.  $(X_t)$  and  $(Y_t)$  are dual with parameter  $q \in \mathbb{R} \setminus \{1\}$  if and only if

$$a' = a + q\gamma, \quad b^{a'} - r' = b^a - r, \quad b^{c'} = b^c + \gamma, \quad c' - r' = c - r - (2+q)\gamma, \quad d' + r' = d + r + \gamma,$$

$$\text{where } \gamma = (2a + (1+q)b^a + qb^c + c - (1-q)r)/(1-q).$$

2.  $(X_t)$  is self-dual with parameter  $q$  if and only if  $q = (r - 2a - b^a - c)/(b^a + b^c + r)$ .

*Proof.* Eq. (11) gives a necessary and sufficient condition for  $q$ -duality of two processes on the same graph  $\Lambda$  with infinitesimal generators  $\bar{G}$  and  $\bar{G}'$  of the form (3), see [S00]. The rates  $\bar{a}, \dots, \bar{d}$  for the generator  $\bar{G}$  are obtained from the rate parameters  $a, \dots, r$  of the basic mechanisms via Eq. (10); similarly for the primed variables. Plugging the expression of  $\bar{a}$  in terms of  $a$ , etc., into Eq. (11) proves the first claim. For the second claim, we note that  $(X_t)$  is self-dual if and only if  $\gamma = 0$ , which determines  $q$  as a function of  $a, \dots, r$ .  $\square$

Now we turn to the resampling-selection process. Fix  $r, s, m > 0$ , set  $a_N = c_N = b_N^a = 0$ , and choose rates  $b_N^c \rightarrow s > 0$ ,  $d_N \rightarrow m > 0$ , and  $r_N/N \rightarrow r > 0$  as  $N \rightarrow \infty$ . Fix  $t > 0$  and let  $(X_\tau^N)_{0 \leq \tau \leq t}$ ,  $(Y_\tau^N)_{0 \leq \tau \leq t}$  be processes constructed from our basic mechanisms with the same rates  $b_N^c$ ,  $d_N$  and  $r_N$ . As explained before Definition 1.1, we can construct these processes on a common probability space, using the same driving Poisson processes;  $(X_\tau^N)$  uses them in the forward time direction,  $(Y_\tau^N)$  uses them backwards. The discrete rescaled processes  $(|X_\tau^N|/N)_{0 \leq \tau \leq t}$  and  $(|Y_\tau^N|/N)_{0 \leq \tau \leq t}$  both have formal generator

$$\begin{aligned} G_N f\left(\frac{k}{N}\right) &= \frac{r_N}{N} k(N-k) \left( f\left(\frac{k+1}{N}\right) + f\left(\frac{k-1}{N}\right) - 2f\left(\frac{k}{N}\right) \right) \\ &\quad + \frac{b_N^c}{N} k(N-k) \left( f\left(\frac{k+1}{N}\right) - f\left(\frac{k}{N}\right) \right) \\ &\quad + \frac{d_N}{N} (k(N-k) + k(k-1)) \left( f\left(\frac{k-1}{N}\right) - f\left(\frac{k}{N}\right) \right). \end{aligned} \quad (12)$$

For  $N \rightarrow \infty$  and  $k/N \rightarrow x$ , this converges to

$$Gf(x) = rx(1-x)f''(x) + sx(1-x)f'(x) - mxf'(x),$$

which is the generator of the diffusion (9), and one can show that the rescaled processes converge to two resampling-selection processes  $(X_\tau)$  and  $(Y_\tau)$ . Now, by Proposition 3.1,  $(X_\tau^N)$  and  $(Y_\tau^N)$  are dual with respect to  $q_N = r_N/(b_N^c + r_N)$ , and by our assumptions on the rates, we have  $N(q_N - 1) \rightarrow -s/r$ . Proposition 1.6 therefore yields

$$\mathbb{E}_x \left[ e^{-(s/r)X_t y} \right] = \mathbb{E}_y \left[ e^{-(s/r)x Y_t} \right],$$

which is the self-duality of the resampling-selection process proven in [AS05]. Thus we have provided a pathwise construction of this self-duality, and we have shown that the self-duality can be obtained by rescaling dual interacting particle systems.

We should note that the latter fact was shown by Swart [Sw06]. His argument, however, starts from *independent* discrete processes  $(X_t^N)$  and  $(Y_t^N)$ , see the proof of Proposition 6 in [Sw06]. In contrast, our

pathwise construction provides a non-trivial coupling of (stopped) underlying discrete processes, which might be of interest in some contexts.

To conclude, we mention that the duality relation can be understood from the following computation. For simplicity, we drop the  $N$ -dependence in the notation. Suppose that at a given time  $\tau$ , there is an arrow from  $i$  to  $j$ , but we do not know of which type it is. The duality parameter  $q = r/(b+r)$  is nothing else but the (conditional) probability that this arrow is of the resampling type. We may think of the arrow as a randomized mechanism with transition matrix

$$P(x, x') = q1_{x'=f^R(x)} + (1-q)1_{x'=f^{BC}(x)},$$

and the self-duality is explained by the relation

$$\forall x, y \in \{0, 1\}^2 : \sum_{x' \in \{0, 1\}^2} P(x, x') q^{|x' \wedge y|} = \sum_{y' \in \{0, 1\}^2} P(y, y') q^{|x \wedge y'|}.$$

## Acknowledgements

Both authors wish to thank the Hausdorff Research Institute for Mathematics in Bonn for hospitality, and Matthias Hammer for many useful comments.

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